DOI: 10.20472/BM.2017.5.2.006

BUSINESS CYCLES WITH PROGRESSIVE INCOME TAXATION

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Abstract:

The paper generalizes the two-sector growth model for a small-open economy with endogenous wealth accumulation and public goods proposed by Zhang (2016). The main generalization is to allow all the time-independent parameters in Zhang's model to be time-dependent. The generalization makes it possible to take any exogenous changes in any parameter at any time into account. The model is constructed in a dynamic general equilibrium framework with heterogeneous households and progressive income taxation. We simulate the model to demonstrate existence of equilibrium and motion of the dynamic system. We demonstrate how business cycles appear in the economic system when it is confronted with different exogenous periodic perturbations.

Keywords:

business cycle; progressive income taxation; wealth accumulation; Walrasian theory; endogenous labor supply

JEL Classification: E32, H24, J20

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Citation:

WEI-BIN ZHANG (2017). Business Cycles with Progressive Income Taxation. International Journal of Business and Management, Vol. V(2), pp. 78-95., 10.20472/BM.2017.5.2.006

1. Introduction

Business cycles and economic fluctuations are well-recoded phenomena. Many studies confirm existence of business cycles in empirical studies and theoretical models (Zhang, 1991, 2005, 2006; Lorenz, 1993; Chiarella and Flaschel, 2000; Shone, 2002; Gandolfo, 2005; Puu, 2011). The theory of business cycles is basically developed by applying nonlinear dynamic analysis in different economic systems. Nevertheless, it can be shown that there are many issues to be examined with regard to theoretically identifying periodic oscillations and f aperiodic fluctuations due to dynamic interdependence between economic growth, economic structural change and wealth accumulation in a heterogeneous-household economy. This paper shows existence of business cycles by generalizing the small-open two-sector growth model with endogenous wealth and public goods proposed by Zhang (2016). We allow all the time-independent parameters in Zhang's model to be time-dependent. By the generalization we can take any exogenous changes in any parameter at any time into account. The model is developed in a dynamic general equilibrium framework with heterogeneous households and progressive income taxation. We simulate the model to demonstrate existence of equilibrium and motion of the dynamic system. We demonstrate how the system reacts to exogenous periodic perturbations.

The model by Zhang (2016) is based on the Walrasian general equilibrium theory of pure exchange and production economies (e.g., Walras, 1874; Arrow and Debreu, 1954; Gale, 1955; Debreu, 1959; McKenzie, 1959; Arrow and Hahn, 1971; Arrow, 1974; Mas-Colell *et al.*, 1995; Montesano, 2008; Impicciatore et al., 2012). Zhang's extends the traditional theory by allowing endogenous wealth and endogenous public goods with nonlinear taxation. The wealth and production side of the model are based on the neoclassical growth theory (e.g., Solow, 1956; and Burmeister and Dobell, 1970). It examines the dynamic relationship between government's spending and private consumption, following the general equilibrium framework (Barro, 1990). It models the economic growth "with two prevalent features observed in developed economies: progressive income taxation together with utility-generating public spending", Chen and Guo (2014: 174). Although there are many studies on the topic (e.g., Barro and Sala-i-Martin, 1995; Futagami *et al*., 1993; Glomm and Ravikumar, 1994, 1997; Agénor, 2011; Baier and Glomm, 2001; Palivos *et al*., 2003; Greiner, 2007; Hu *et al*., 2008; Kamiguchi and Tamai, 2011), Zhang's model deviates from the traditional Ramsey approach by applying Zhang's utility function (Zhang, 1993). This paper is organized as follows. Section 2 introduces the basic model with wealth accumulation, endogenous time distribution between labor and leisure, and public services for a small open economy with progressive income taxation and heterogeneous households. Section 3 simulates the motion of the economic system. Section 4 demonstrates the effects of changes in some parameters on the economic system. Section 5 concludes the study.

2. The basic model

This section generalizes the small-open two-sector growth model with endogenous wealth and public goods proposed by Zhang (2016) by allowing all the time-independent parameters to be time-dependent. By the generalization we can take any exogenous changes in any parameter at any time into account. This makes Zhang's analysis much more robust. The open economy under consideration can import goods and borrow resources from the rest of the world or exports goods and lend resources abroad. There is a single good, called industrial good, in the world economy and the price of the industrial good is unity. The rate of interest $r^*(t)$ in time t is fixed in international market. Capital depreciates at a rate, $\delta_k(t)$. The economy has one production sector and one public goods sector. The production sector follows the standard one-sector growth model (Burmeister and Dobell, 1970; Barro and Sala-i-Martin, 1995). The production sector uses labor and capital inputs to produce good. The population is fully employed in either of the two sectors. The assets of the economy are owned by households. The households' incomes are used to consume and save. Exchanges take place in perfectly competitive markets. Factor markets work well and factors are fully utilized at every moment. Saving is undertaken only by households. The population is classified into J groups according to human capital and preference. We use subscripts j to stand for group j, $j = 1, ..., J$. Group $j's$ population is denoted by $N_j(t)$. We use $T_j(t)$ to stand for the work time of household j and $N(t)$ for the flow of labor services used at time t for production. We have $N(t)$ as follows

$$N(t) = \sum_{j=1}^{J} h_j(t) N_j(t) T_j(t), \tag{1}$$

where $h_j(t)$ is the level of human capital of group j.

The production sector

We specify the production function as follows

$$F(t) = A(t) K_i^{\alpha_i(t)}(t) N_i^{\beta_i(t)}(t), \ \ \alpha_i(t) + \beta_i(t) = 1, \ \ \alpha_i(t), \ \beta_i(t) > 0, \tag{2}$$

Where $A(t)$, $\alpha_i(t)$ and $\beta_i(t)$ are time-dependent parameters. To analyze progressive income taxation, like Chen and Guo (2014) we omit possible effects of public goods on productivity. The marginal conditions are

$$r^{*}(t)=\frac{\alpha_i(t)F(t)}{K_i(t)}-\delta_k\,,\quad w(t)=\frac{\beta_i(t)F(t)}{N_i(t)},\quad w_j(t)=h_j(t)w(t), \tag{3}$$

where $w_j(t)$ is the wage rate per unit of time for group j.

The disposable income and budget constraint

Let $\bar{k}_j(t)$ stand for the capital stock owned by the representative household of group j. When there is no taxation on the household's current income the household income from the interest payment and the wage payments is

$$y_{0j}(t)=r^{*}(t)\bar{k}_j(t)+w_j(t)T_j(t). \tag{4}$$

The progressive tax rate $\tau_{hj}(t)$ is assumed to be a function of $y_{0j}(t)$ as follows

$$\tau_{hj}(t)=\tau_{0j}(t)+\tau_j(t)y_{0j}^{a_j(t)}(t),\;\; 1>\tau_{0j}(t),\, a_j(t)>0,\, \tau_j(t)>0. \tag{5}$$

In the case of $a_j>0$ the tax rate is increased as the per capita income is increased. The tax schedule is said to be progressive. In the case of $a_j=0$ the tax schedule is called flat. Many studies assume a constant tax rate of income or a flat consumption tax (e.g., Cazzavillan, 1996; Zhang, 2000; Raurich, 2003; Fernández *et al*. 2004; Chen, 2006; Guo and Harrison, 2008). This study assumes $a_j(t)$ to be dependent on time. The representative household's current income $y_j(t)$ is

$$y_j(t)=\bar{\tau}_{hj}(t)y_{0j}(t), \tag{6}$$

where $\bar{\tau}_{hj}(t)=1-\tau_{hj}(t)$. The per household disposable income of the household is defined as the sum of the current income and the wealth available for purchasing consumption goods and saving

$$\hat{y}_j(t)=y_j(t)+\bar{k}_j(t).$$

The disposable income is used for saving and consumption. At each point in time, the household would distribute the total available budget between savings $s_j(t)$ and consumption of goods $c_j(t)$. The budget constraint is given by

$$c_j(t) + s_j(t) = \hat{y}_j(t).$$

Let $\bar{T}_j(t)$ stand for the leisure time at time t and the (fixed) available time for work and leisure by T_0. The time constraint is expressed by

$$T_j(t) + \bar{T}_j(t) = T_0 .$$

Insert this time constraint in the disposable income

$$\hat{y}_j(t) = \bar{y}_j(t) - \bar{\tau}_{hj}(t) w_j(t) \bar{T}_j(t), \tag{7}$$

where

$$\bar{y}_j(t) \equiv \bar{k}_j(t) + \bar{\tau}_{hj}(t) r^* \bar{k}(t) + \bar{\tau}_{hj}(t) w_j(t) T_0 .$$

Insert (5) in the budget constraint

$$\bar{\tau}_{hj}(t) w_j(t) \bar{T}_j(t) + c_j(t) + s_j(t) = \bar{y}_j(t). \tag{8}$$

The utility function

We assume that utility level $U_j(t)$ is dependent on the leisure times, the consumption level of commodity, and savings as follows

$$U_j(t) = u_j(G(t),t) \bar{T}_j^{\sigma_{0j}(t)}(t) c_j^{\xi_{0j}(t)}(t) s_j^{\lambda_{0j}(t)}(t), \;\; \sigma_{0j}(t),\, \xi_{0j}(t),\, \lambda_{0j}(t) > 0,$$

where $u_j(t)$ is dependent on the level of public services, $\sigma_{0j}(t)$, $\xi_{0j}(t)$ and $\lambda_{0j}(t)$ are called respectively the propensities to stay at home, to consume good and to hold wealth in time t.

The household's optimal behaviour

Maximizing U_j subject to budget constraint (8) yields

$$\bar{\tau}_{hj}(t)w_j(t)\bar{T}_j(t) = \sigma_j(t)\bar{y}_j(t), \quad c_j(t) = \xi_j(t)\bar{y}_j(t), \quad s_j(t) = \lambda_j(t)\bar{y}_j(t), \tag{9}$$

where

$$\rho_j(t) \equiv \frac{1}{\sigma_{0j}(t) + \xi_{0j}(t) + \lambda_{0j}(t)}, \quad \sigma_j(t) \equiv \rho_j(t)\sigma_{0j}(t), \quad \xi_j(t) \equiv \rho_j(t)\xi_{0j}(t), \quad \lambda_j(t) \equiv \rho_j(t)\lambda_{0j}(t).$$

The wealth accumulation

According to the definitions of $s_j(t)$ and $\bar{k}_j(t)$, the change in the household's wealth is given by

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t). \tag{10}$$

This equation simply means that change in the wealth is equal to saving minus dissaving.

The public sector

The public sector is financially supported by the government's tax income. The public sector has the following income

$$I_p(t) = \sum_{j=11}^{J} \left[r^* \bar{k}_j(t) + w_j(t)T_j(t)\right]\tau_{hj}(t)N_j. \tag{11}$$

We assume that the public sector supplies public goods by utilizing capital, $K_p(t)$, and labor force, $N_p(t)$, as follows

$$G(t) = A_p K_p^{\alpha_p}(t) N_p^{\beta_p}(t), \quad A_p, \alpha_p, \beta_p > 0, \quad \alpha_p + \beta_p = 1.$$

The capital stocks and workers employed by the public sector are paid at the same rates that the private sector pays the services of these factors. The budget constraint of the public sector is

$$w(t) N_p(t) + \left(r^*(t) + \delta_k(t)\right) K_p(t) = I_p(t). \tag{12}$$

The public sector behaves effectively in the sense that it uses the available resource to maximize public services. Maximizing public services under the budget constraint yields

$$\frac{\alpha_p N_p(t)}{\beta_p K_p(t)} = \frac{r^*(t) + \delta_k(t)}{w(t)}. \tag{13}$$

The factors are fully employed

The capital stock used by the country $K(t)$ and the national labor force $N(t)$ are fully employed

$$K_i(t) + K_p(t) = K(t), \quad N_i(t) + N_p(t) = N(t). \tag{14}$$

The national wealth

The national wealth $\bar{K}(t)$ is equal to the sum of the wealth owned by all the households in the country

$$\bar{K}(t) = \sum_{j=1}^{J} \bar{k}_j(t) N_j. \tag{15}$$

We thus built the model with progressive income taxation. In the rest of the paper, we will examine properties of the model and see how changes in different parameters will affect the economic system.

3 The behavior of the model

This section studies dynamics of the model. First, we show that the motion of the entire economic system can be described by J differential equations with the tax rates as the variables. The following lemma is checked in the Appendix.

Lemma

The dynamics of the economic system is governed by the following J differential equations with the J tax rates on the J groups as the variables

$$\dot{\tau}_{hj}(t) = \left(\Lambda_j\left(\left(\tau_{hj}(t)\right), t\right) - \frac{\partial f_j\left(\left(\tau_{hj}(t)\right), t\right)}{\partial t} \right) \left(\frac{\partial f_j\left(\left(\tau_{hj}(t)\right), t\right)}{\partial \tau_{hj}} \right)^{-1}, \quad j = 1, ..., J \tag{16}$$

where the right-hand sides of (15) are functions of $\tau_{hj}(t)$ defined in the appendix. The values of all the other variables are uniquely determined as functions of $\tau_{hj}(t)$ at any point in time by the following procedure: $w(t)$ by (A3) $\rightarrow w_j(t) = h_j(t) w(t)$ $\rightarrow$ $N_p(t)$ by (A11) $\rightarrow$ $K_p(t)$ by (A1) $\rightarrow$ $I_p(t) = w(t) N_p(t) / \beta_p(t)$ $\rightarrow \bar{k}_j(t)$ by (A9) $\rightarrow$ $N(t)$ by (A6) $\rightarrow$ $K_i(t)$ by (A10) $\rightarrow$ $N_i(t)$ by (A1) $\rightarrow$ $K(t)$ by (A12) $\rightarrow$ $\bar{K}(t)$ by (15) $\rightarrow$ $y_{0j}(t)$ by (A8) $\rightarrow$ $T_j(t)$ by (A5) $\rightarrow$ $\bar{T}_j(t) = T_0 - T_j(t)$ $\rightarrow \bar{y}_j(t)$ by the definition $\rightarrow$ $F(t)$ by (A2) $\rightarrow G(t)$ by the definition $\rightarrow$ $c_j(t)$ and $s_j(t)$ by (A7).

Following Zhang (2016), we simulate the model when the parameters are constant. The next section examines behavior of the system when we allow parameters to be time-dependent. we consider that the economy has 3 groups of the population. We specify the parameter values as follows

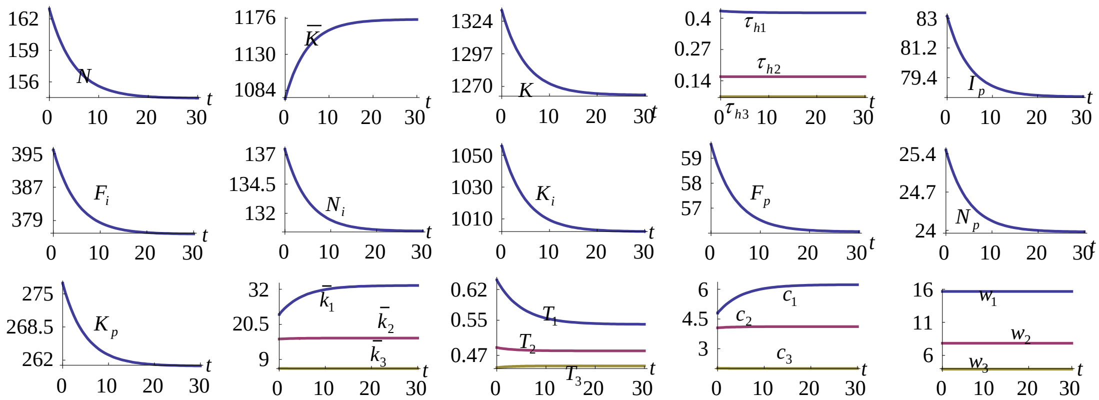
$$\begin{gathered} N_1 = 10, \; N_2 = 30, \; N_3 = 60, \; T_0 = 1, \; r^* = 0.07, \; \alpha_i = 0.32, \; \alpha_p = 0.4, \; A = 1.5, \\ A_p = 0.9, \; h_1 = 8, \; h_2 = 4, \; h_3 = 2, \; \xi_{01} = 0.15, \; \lambda_{01} = 0.8, \; \sigma_{01} = 0.1, \; \xi_{02} = 0.18, \\ \lambda_{02} = 0.7, \; \sigma_{02} = 0.15, \; \xi_{03} = 0.6, \; \lambda_{01} = 0.2, \; \sigma_{01} = 0.2, \; \tau_{01} = 0.06, \; \tau_1 = 0.2, \\ a_1 = 0.25, \; \tau_{02} = 0.03, \; \tau_2 = 0.1, \; a_2 = 0.25, \; \tau_{03} = 0.02, \; \tau_3 = 0.05, \; a_3 = 0.1, \\ \delta_k = 0.05. \end{gathered} \tag{14}$$

The rate of interest is 0.07. Group 1's propensity to save and human capital are the highest, group 2's are the next, and group 3's are the lowest. The total productivities of the two sectors are specified at 1.5 and 0.9, respectively. With regard to the technological parameters, what are important in our study are their relative values. We simulate the model the following initial conditions

$$\tau_{h1}(0) = 0.43, \quad \tau_{h2}(0) = 0.157, \quad \tau_{h1}(0) = 0.074.$$

The changes of the variables over time are plotted in Figure 1. The tax rates change slightly over time. Groups 1 and 2 reduce their working hours and groups 3 increased its working hours. The consumption levels of groups 1 and 2 are increased and the consumption level of groups 3 is slightly reduced. The production levels of the two sectors are reduced over time. The trade balance is improved. The nation uses less capital and holds more wealth. The national labor force falls.

Figure 1. The Motion of the Dynamic System



The equilibrium values of the variables are given as follows

$$w_1 = 15.67, \quad w_2 = 7.83, \quad w_3 = 3.92, \quad N = 154.4, \quad K = 1262.8, \quad \bar{K} = 1174.6, \quad E = -6.17,$$
$$\tau_{h1} = 0.423, \quad \tau_{h2} = 0.157, \quad \tau_{h3} = 0.074, \quad I_p = 78.25, \quad F_i = 375.8, \quad F_p = 56.05,$$
$$N_i = 130.5, \quad N_p = 23.97, \quad K_i = 1002, \quad K_p = 260.82, \quad \bar{k}_1 = 33.3, \quad \bar{k}_2 = 16, \quad \bar{k}_3 = 6.03, \tag{15}$$
$$T_1 = 0.54, \quad T_2 = 0.48, \quad T_3 = 0.45, \quad c_1 = 6.24, \quad c_2 = 4.12, \quad c_3 = 2.$$

The three eigenvalues are

$$-0.36, \quad -0.278, \quad -0.197.$$

The equilibrium point is stable. This result is important as it allows us to effectively conduct comparative dynamic analysis.

4. Comparative Dynamic Analysis

We plotted the motion of the economic system in the previous section. We now study effects of changes in some parameters on the motion of the economic system. Zhang (2016) shows how the system reacts to a once-for-all change in parameters. This section shows how the system reacts to time-dependent changes in parameters. For convenience we consider the parameters in (14) as the long-term average values. We make small perturbations around these long-term values. As we explicitly provided the procedure to simulate the motion, it is straightforward to make comparative dynamic analysis.

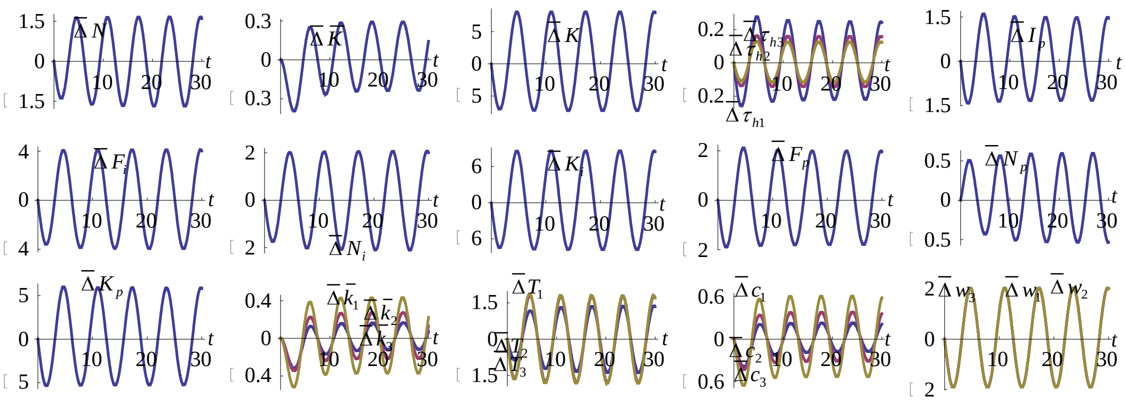
4.1. Fluctuations in the rate of interest in the global market

Relations between exogenous rate of interest and economic growth are emphasized in the literature of small-open economic growth theory. Nevertheless, it is difficult to examine effects of changes in internationally influenced cost of capital when the cost is fluctuating over time. To follow the dynamics over time we need a genuine dynamic model and be able to follow paths of motion. We can do this as we can simulate the model. We now examine what will happen if the rate of interest experiences the following fluctuations

$$r^*(t) = 0.07 + 0.005\sin(t).$$

The impacts on the variables are plotted in Figure 2. In the plots, a symbol $\bar{\Delta}$ stands for the change rate of the variable in percentage due to changes in the parameter value. The periodic exogenous fluctuations lead to economic business cycles. The national force, the industrial sector's output and labor input experience fluctuations with large amplitudes. It should be noted that the amplitude of capital stocks employed by the national economy is larger than that of national wealth. The wage rates oscillate similarly. The tax income and factor inputs of the public sector fluctuate around their long-run values.

Figure 2. Fluctuations in the Rate of Interest in the Global Market



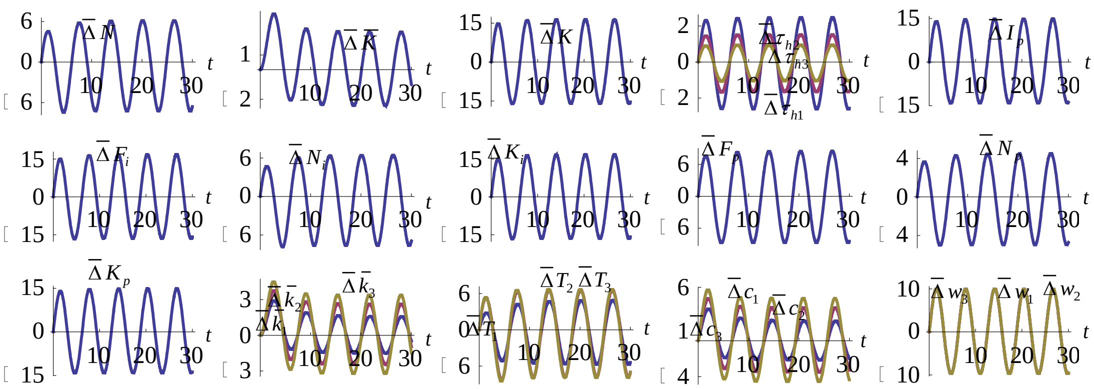
4.2. Fluctuations in the total factor productivity of the industrial sector

Exogenous oscillations in the total factor productivity of the industrial sector are specified in the following way

$$A(t) = 1.5 + 0.1\sin(t).$$

The simulation results are plotted in Figure 3. From the figure we see that the changes in the parameter have great impacts on the variables. The periodic oscillations lead to periodic business cycles. The amplitude of fluctuations in the national wealth is smaller than the capital stock employed by the economy. The oscillations in the productivity causes fluctuations in the total wealth, the output levels and two input factors of the two sectors. The changes in the parameter also cause oscillations in the wage rates, consumption levels and time distribution.

Figure 3. Fluctuations in the Total Factor Productivity of the Industrial Sector



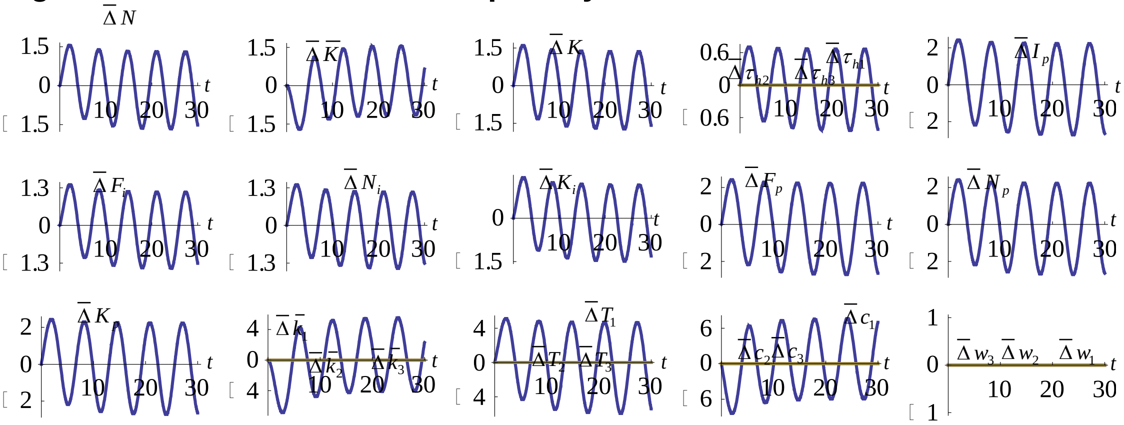
4.3. Fluctuations in the propensity to save

There are many studies on relationships between economic growth and propensity to save. As our model is developed in a general equilibrium framework, allowing all the parameters to be exogenously changed in any manner, we can study effects of changes in the propensity to save. We allow exogenous oscillations in group 1’s propensity to save as follows

$$\lambda_{01}(t) = 0.8 + 0.05\sin(t).$$

The simulation results are plotted in Figure 3. From the figure we see that the changes in the parameter have great impacts on the variables. The periodic oscillations lead to periodic business cycles. The amplitude of fluctuations in the national wealth is smaller than the capital stock employed by the economy. The oscillations in the productivity causes fluctuations in the total wealth, the output levels and two input factors of the two sectors. The changes in the parameter also cause oscillations in the wage rates, consumption levels and time distribution.

Figure 4. Fluctuations in the Propensity to Save



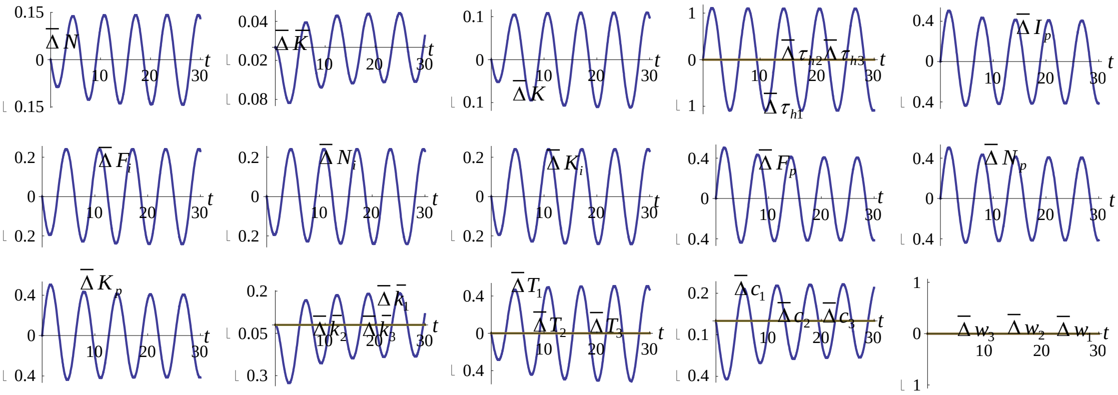
4.4. Fluctuations in the tax rate

We allow exogenous oscillations in group 1's taxation function as follows

$$\tau_{01}(t) = 0.065 + 0.005\sin(t).$$

The simulation results are plotted in Figure 5. From the figure we see that the changes in the parameter have periodic impacts on the variables. The periodic oscillations lead to periodic business cycles. Like in the previous case, the amplitude of fluctuations in the national wealth is smaller than the capital stock employed by the economy. The oscillations in taxation cause fluctuations in the total wealth, the output levels and two input factors of the two sectors. The wage rates are not affected. Group 1's work time, consumption level and wealth experience oscillations, but the other two groups' corresponding variables are not affected. We see that the changes in the national variables are due to the changes in group 1' economic conditions.

Figure 5. Fluctuations in the Tax Rate



5. Concluding Remarks

This paper generalized the small-open two-sector growth model with endogenous wealth and public goods proposed by Zhang (2016). We allowed all the time-independent parameters to be time-dependent. By the generalization we can take any exogenous changes in any parameter at any time into account. This makes Zhang's analysis much more robust. The open economy under consideration can import goods and borrow resources from the rest of the world or exports goods and lend resources abroad. The model is developed in a dynamic general equilibrium framework with heterogeneous households and progressive income taxation. The public sector is financially supported by the tax income. The model describes dynamic interactions between economic growth, wealth accumulation, and progressive income taxation in an internationally open environment. The model is valid for any number of types of households. We simulated the model to demonstrate existence of equilibrium and motion of the dynamic system. We demonstrated how the system reacts to exogenous periodic perturbations. The model may be generalized in multiple directions. We may extend the model to multiple countries. It is also possible to extend the one production sector to multiple sectors. We may study different tax policies for different sectors. Our model is built in an integrated Walrasian-general equilibrium and neoclassical-growth theory. As there is a large number of the literature in each of the theories, we may generalize our model basing on the literatures.

Appendix: Proving the Lemma

From (3) and (11) we have

$$z \equiv \frac{r^* + \delta_k}{w} = \frac{N_i}{\bar{\beta}_i K_i} = \frac{N_p}{\bar{\beta}_p K_p}, \tag{A1}$$

where $\bar{\beta}_j \equiv \beta_j / \alpha_j$, $j = i, p$. For simplicity, we omit time in expressions when the omission will not cause confusion. From (A1) and the production function, we have

$$F = A \bar{\beta}_i^{\beta_i} K_i z^{\beta_i} . \tag{A2}$$

From (3) and (A2), we obtain

$$z = \left(\frac{r^* + \delta_k}{A \alpha_i} \right)^{1/\beta_i} \frac{1}{\bar{\beta}_i}, \quad w = \frac{A_w}{z^{\alpha_i}}, \tag{A3}$$

where $A_w \equiv A \beta_i / \bar{\beta}_i^{\alpha_i}$. Insert (A3) in the definition of $\bar{y}_j$

$$\bar{y}_j = \left(1 + r^* \bar{\tau}_{hj}\right) \bar{k}_j + \bar{\tau}_{hj} w_j T_0 . \tag{A4}$$

Insert (9) in $T_j + \bar{T}_j = T_0$

$$T_j = \tilde{\alpha}_j - \frac{\left(1 + r^* \bar{\tau}_{hj}\right) \tilde{\sigma}_j \bar{k}_j}{\bar{\tau}_{hj}}, \tag{A5}$$

where we use (A4) and

$$\tilde{\alpha}_j \equiv \left(1 - \sigma_j\right) T_0, \quad \tilde{\sigma}_j \equiv \frac{\sigma_j}{w_j}.$$

From (1) and (A5) we have the labor force as follows

$$N = h - \sum_{j=1}^{J} \frac{\left(1 + r^* \bar{\tau}_{hj}\right) \bar{\sigma}_j \bar{k}_j}{\bar{\tau}_{hj}}, \tag{A6}$$

where

$$h \equiv \sum_{j=1}^{J} \tilde{\alpha}_j \, h_j \, N_j \, , \quad \bar{\sigma}_j \equiv h_j \, N_j \, \tilde{\sigma}_j \, .$$

From (4) and (A5), we have

$$y_{0j} = \left[r^* - \frac{\left(1 + r^* \, \bar{\tau}_{hj}\right) \tilde{\sigma}_j \, w_j}{\bar{\tau}_{hj}} \right] \bar{k}_j + w_j \, \tilde{\alpha}_j \, . \tag{A7}$$

From (5) we have

$$y_{0j} = \left(\frac{\tau_{hj} - \tau_{0j}}{\tau_j} \right)^{1/a_j} . \tag{A8}$$

From (A7) and (A8) we have

$$\bar{k}_j = f_j\left(\tau_{hj}, t\right) \equiv \left[\left(\frac{\tau_{hj} - \tau_{0j}}{\tau_j} \right)^{1/a_j} - w_j \, \tilde{\alpha}_j \right] \left[r^* - \frac{\left(1 + r^* \, \bar{\tau}_{hj}\right) \tilde{\sigma}_j \, w_j}{\bar{\tau}_{hj}} \right]^{-1} . \tag{A9}$$

Insert (A1) in (14)

$$\bar{\beta}_i \, K_i + \bar{\beta}_p \, K_p = \frac{N}{z} . \tag{A10}$$

Insert $w N_p = \beta_p \, I_p$ in (11)

$$N_p = \frac{\beta_p}{w} \sum_{j=11}^{J} \tau_{hj} \, N_j \left(\frac{\tau_{hj} - \tau_{0j}}{\tau_j} \right)^{1/a_j} , \tag{A11}$$

where we also use (A8). We can see that the variables can be expressed as functions of (τ_{hj}) as follows: z by (A3) $\rightarrow$ w by (A3) $\rightarrow w_j = h_j w$ $\rightarrow$ N_p by (A11) $\rightarrow$ K_p by (A1) $\rightarrow$ $I_p = w N_p / \beta_p \rightarrow \bar{k}_j$ by (A9) $\rightarrow$ N by (A6) $\rightarrow$ K_i by (A10) $\rightarrow$ N_i by (A1) $\rightarrow$ K by (A12) $\rightarrow$ $\bar{K}$ by (15) $\rightarrow$ E by (14) $\rightarrow$ y_0 by (A8) $\rightarrow$ T_j by (A5) $\rightarrow$ $\bar{T}_j = T_0 - T_j \rightarrow \bar{y}_j$ by the definition $\rightarrow$ F by (A2) $\rightarrow G$ by the definition $\rightarrow$ c_j and s_j by (A7).

Taking derivatives of (A9) with respect to t yields

$$\dot{\bar{k}}_j = \frac{\partial f_j}{\partial \tau_{hj}} \dot{\tau}_{hj} + \frac{\partial f_j}{\partial t}. \tag{A12}$$

We do not provide the explicit expression of $\partial f_j / \partial \tau_{hj}$ as it is tedious. From (8) we have

$$\dot{\bar{k}}_j = \Lambda_j(\tau_{hj}) \equiv s_j - f_j. \tag{A13}$$

From (A12) and (A13), we solve

$$\dot{\tau}_{hj} = \left(\Lambda_j - \frac{\partial f_j}{\partial t} \right) \left(\frac{\partial f_j}{\partial \tau_{hj}} \right)^{-1}. \tag{A14}$$

In summary, we proved the lemma.

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