Zero lower bounds and a Stackelberg problem:  
Stochastic analysis of unconventional monetary policy

Koiti Yano*

September 10, 2008: first draft
August 1, 2013: this draft

Abstract

There exist difficulties in escaping from stagnation and/or deflation if the economy hits zero lower bounds on short-term nominal interest rates because the central bank cannot stimulate the economy further using rate cuts. How can it escape from them? Answering the question, we extend a closed-looped solution of a Stackelberg problem (also known as a Ramsey problem) by introducing zero lower bounds. Herein, we formulate a constrained Stackelberg problem and derive a solution for it. In our extension, we found that the discounted Lyapunov equation is necessary to obtain the shadow price of the economy that hits the zero lower bounds. Herein we apply our method to new Keynesian models with zero lower bounds. Our simulation shows that committing to mild inflation engenders positive effects on the economy and that managing inflation expectations is necessary to escape from the bounds.

Keywords: discounted Lyapunov equation, zero interest rate, inflation target, managing expectation

JEL Classification Codes: E32, E52, C54

1 Introduction

After the financial collapse in 2008, Zero Lower Bounds (ZLBs) on short-term nominal interest rates have become a practical issue for central banking because some advanced economies (US, UK, and Japan) hit them. If an economy hits ZLBs, the central bank of it cannot stimulate it further using rate cuts. In Figure 1, we show the 3-Month US government Treasury Bill rates. This figure shows that conventional monetary policy, which stimulates the economy by cutting short-term nominal interest rates, is not applicable to the situation. Consequently, unconventional monetary policy becomes a hot topic in monetary theory and policy.

*Faculty of Economics, Komazawa University. E-mail: koiti@komazawa-u.ac.jp, koiti.yano@gmail.com.
Unconventional monetary policy is roughly categorized into Zero Interest Rate Policy (ZIRP), Quantitative Easing (QE), Exchange Rate Intervention (ERI), Inflation Targeting (IT), and Price Level Targeting (PLT). Their features are summarized:

- **ZIRP**: Committing to keep zero or nearly zero interest rates in extended periods. Such a policy commitment was conducted by the Bank of Japan in 1999.

- **QE**: Purchasing unconventional assets in financial markets. This approach also shows balance sheet expansion of the central bank. It was conducted by BOJ in the 2000s and the US Federal Reserve and the Bank of England after 2008.

- **ERI** (also known as a “foolproof way”): Intervening in the exchange rate markets or pegging exchange rates, proposed by McCallum (2000), Svensson (2001), and Coenen and Wieland (2003)

- **IT**: Committing to a explicit target for inflation, as proposed in an earlier report Krugman (1998).

- **PLT**: Committing to target a certain price level, as proposed in an earlier report Eggertsson and Woodford (2003).

Irrespective of the variety of these unconventional monetary policies, the common feature among

---

1Of course, fiscal policy might be an effective approach when the economy hits ZLBs. That is, however, beyond the scope of the discussion presented in this paper.
them is that the central bank must conduct a ZIRP during them. Consequently, the most important emphasis of our manuscript is the effects of ZIRP.

The seminal work by Krugman (1998) points out that managing inflation expectations plays a central role in escape from ZLBs. His paper has spurred the publication of many related papers, such as those listed below: Eggertsson and Woodford (2003), Jung, Teranishi, and Watanabe (2005), Christiano (2004), Auerbach and Obstfeld (2005), Sugo and Teranishi (2005), Adam and Billi (2006), Braun and Waki (2006), Eggertsson (2008), and Nakov (2008). 2 In these studies (except Adam and Billi (2006), Nakov (2008), Takamura, Watanabe, and Kudo (2006)), Perfect Foresight or Non-Stochastic Environment is assumed. PF or NSE are not consistent with the rational expectations hypothesis in DSGE modeling. However, some studies are useful to propose new methods, which are consistent with the rational expectations hypothesis. Adam and Billi (2006) and Nakov (2008) propose a new simulation method, which is consistent with RE, based on Marcet and Marimon (1998). Takamura, Watanabe, and Kudo (2006) propose a linear quadratic approach based on Soderlind (1999). These prominent methods based on RE, however, are very complex to simulate large or medium scale DSGE models. Therefore, we must develop a novel and simple method.

In previous studies of conventional monetary policy, the monetary authority is assumed to determine the nominal interest rate according to the Taylor rule with a ZLB. However, it is generally complicated to deal with DSGE models with the Taylor rule with the ZLB because there exists nonlinearity of models. Moreover, the optimal rule should be derived from an economic optimization problem. Exceptions are Kato and Nishiyama (2005) and Nishiyama (2009). Their method, however, is based on backward-looking models. Consequently, deriving optimal monetary policy rule with ZLBs is necessary. 3

To solve these problems, we extend a closed-looped solution of a Stackelberg problem (also known as a Ramsey problem), proposed by Hansen and Sargent (2003) and Hansen and Sargent (2007), by introducing zero lower bounds. Since the seminal works of Kydland and Prescott (1977), Calvo (1978), Barro and Gordon (1983a), and Barro and Gordon (1983b), it has become widely known that the open-loop solutions of the Stackelberg problem are time-inconsistent. In response to the problem, Hansen and Sargent (2003) devise the closed-looped solution of a Stackelberg problem based on the robust control to analyze economic problems with model uncertainty. In the solution, however, ZLBs are not considered explicitly. Therefore, it is crucial

---


3 Optimal monetary policy rule with a ZLB is derived in Appendix C.
to investigate a Stackelberg solution with ZLBs.  

In seminal works of Kydland and Prescott (1977), Calvo (1978), Barro and Gordon (1983a), and Barro and Gordon (1983b), to analyze the time-inconsistency problem of central banking, the standard intertemporal loss function of the central bank is assumed as follows.

$$
E_0 \sum_{t=0}^{\infty} \delta^t L(y_t, u_t) = \left[ E_0 \sum_{t=0}^{\infty} \delta^t (y_t'Ry_t + u_t'Qu_t) \right],
$$

Therein, $y_t$ is the vector of endogenous variables, $u_t$ is the vector of a control variable, $R$ is a real positive semidefinite matrix, and $Q$ is a real symmetric positive definite matrix. Following these seminal works, we evaluate the optimality of unconventional monetary policies using the loss function.

To analyze the ZIRP effects, we must investigate a Stackelberg problem, which is defined by Ljungqvist and Sargent (2004). For the convenience for readers, we quote their definition.

**Definition 1.1.** Stackelberg Problem (Chapter 18, Ljungqvist and Sargent (2004)): The problem is to minimize Eq. (8) by finding a sequence of decision rules, the time $t$ component of which maps the time $t$ history of the state $z^t$, where it denotes the history of any variable $z$ from 0 to $t$, into the time $t$ decision $u_t$ of the Stackelberg leader. The Stackelberg leader commits to this sequence of decision rules at time 0. The minimization is subject to a given initial condition for $z_0$. However, $x_0$ is to be chosen.

After defining the problem, they argue that the optimal decision rule is history dependent, meaning that $u_t$ depends not only on $z_t$ but also on lags of $z$. The history dependence has two sources: (a) the government’s ability to commit to a sequence of rules at time 0, and (b) the forward-looking behavior of the private sector. To resolve the problem, Hansen and Sargent (2003), Hansen and Sargent (2007), and Ljungqvist and Sargent (2004) propose a closed-loop solution of a Stackelberg problem (a Stackelberg solution).

Although the previous studies of the Stackelberg solution incubate our approach, this approach has a single difference from their Stackelberg problem because we specifically examine the existence of ZLBs. In the ZLB case, the policy interest rate of the central bank is constrained. To solve our problem, it is necessary for us to extend a closed-looped solution of a Stackelberg

---

4 In Ljungqvist and Sargent (2004), they show a simplified Stackelberg solution without model uncertainty. To undertake our specific examination of ZIRP and managing expectations, we abbreviate model uncertainty in this paper. Consequently, our formulation of a Stackelberg problem is based on Ljungqvist and Sargent (2004). The uncertainty problem is the next step of our agenda.

5 A prominent paper by Levine and Pearlman (2008) and Levine and Pearlman (2010) analyzes ZLBs based on Hansen and Sargent (2003). However, the paper specifically examines model uncertainty and does not solve ZLB problems explicitly using the Karush–Kuhn–Tucker theorem and the discounted Lyapunov equation.
problem by introducing zero lower bounds (a constrained Stackelberg solution). In the extension, we found that the discounted Lyapunov equation is necessary to obtain the shadow price of the economy, which hits the zero lower bounds. Additionally, we emphasize that our method is consistent with the rational expectations hypothesis.

As described in this paper, we apply our method to new Keynesian models with ZLBs to evaluate the quantitative effects of zero interest rate policies with committing mild or zero inflation. Our simulation reveals that committing mild inflation causes positive effects on the economy and that managing inflation expectations is necessary to escape from the bounds. Additionally, it is apparent that the economy, which hits ZLBs, is vulnerable to exogenous shocks.

This paper is organized as follows. In section 2, we formulate new Keynesian models with ZLBs and a constrained Stackelberg solution. In section 3, simulations based on the solution are conducted. In section 4, we describe conclusions and our future research agenda.

2 Method

2.1 Model

The early works of Kimball (1995), Roberts (1995), and Yun (1996) fostered the development of many subsequent papers (see McCallum and Nelson (1999), Clarida, Gali, and Gertler (1999), Gali (2002), and related studies are referred therein). Following those earlier studies, the new Keynesian IS curve is

\[
\hat{Y}_t = \frac{h}{1 + h} \hat{Y}_{t-1} + \frac{1}{1 + h} E_t \hat{Y}_{t+1} - \frac{1}{1 + h} E_t [\hat{i}_t - \hat{r}_{t+1} - \hat{r}^\pi_t] + \eta^i_t,
\]

where \( \hat{Y}_t \) denotes an output gap, \( \hat{i}_t \) signifies a nominal interest rate, \( \hat{r}_t \) is an inflation rate, \( \hat{r}^\pi_t \) is the exogenous shock to the economy, and \( \eta^i_t \sim N(0, \sigma_i^2) \) is a random variable. \( \hat{i}_t \) is an instrument of the central bank. Second, the new Keynesian Phillips curve is

\[
\hat{\pi}_t = \frac{1}{1 + \beta} \hat{\pi}_{t-1} + \frac{\beta}{1 + \beta} E_t \hat{\pi}_{t+1} + \kappa \hat{Y}_t + \eta^\pi_t.
\]

where \( \beta \) represents a discount factor, \( \kappa \) denotes a constant, and \( \eta^\pi_t \sim N(0, \sigma^2_\pi) \) is a random variable. Following Eggertsson and Woodford (2003) Jung, Teranishi, and Watanabe (2005), and Sugo and Teranishi (2005), the dynamics of the exogenous shocks is

\[
\hat{r}^\pi_t = \rho \hat{r}^\pi_{t-1} + \eta^\pi_t.
\]

---

6 Deriving the solution is described in Appendix A.

7 See also Woodford (2003), Kato (2006), Gali (2008), and Walsh (2010), and related studies are referred therein. Giordani and Soderlind (2004) and Dennis, Leitemo, and Soderstrom (2009) modify Hansen and Sargent (2003) to apply new Keynesian models.

8 As described in this paper, a hat over a variable signifies the percentage deviation from its steady state value.
where \( \rho_n \) stands for a constant value and \( \eta_t \sim N(0, \sigma_n^2) \) is a random variable. \( \hat{Y}_t, \hat{\pi}_t, \) and \( \hat{r}_t \) are predetermined variables and \( E_t \hat{Y}_{t+1} \) and \( E_t \hat{\pi}_{t+1} \) are non-predetermined variables.

As described in this paper, we presume that there exists a Zero Lower Bound (ZLB) on short-term nominal interest rates:

\[
\hat{i}_t \geq 0. \tag{4}
\]

ZLBs have been practical issues of monetary policy for developmentally advanced economies (e.g., US, UK, and Japan) after 2008 because their short-term nominal interest rates hit the bounds. As described in this paper, we assume that the ZLBs bind from time 0 to \( k \) and that they occur only once in the economy.

For convenience, we designate \( z_t = [\hat{Y}_t, \hat{\pi}_t, \hat{r}_t] \) as the vector of predetermined variables, \( x_t = [E_t \hat{Y}_{t+1}, E_t \hat{\pi}_{t+1}] \) as the vector of non-predetermined variables, \( y_t = [z_t, x_t]^T \), and \( u_t = [\hat{i}_t] \geq 0. \)

Using Eq. (1), (2), and (3), we can construct a structural linear rational expectations model

\[
\tilde{G} y_{t+1} = \tilde{A} y_t + \tilde{B} u_t + \tilde{C} \eta_t, \tag{5}
\]

where \( \tilde{G}, \tilde{A}, \tilde{B} \) are matrices, \( \tilde{C} \) is a diagonal matrix, and \( \eta_t \sim N(0, \Sigma_{sr}^2) \) with the \((5 \times 5)\) identity matrix \( \Sigma_{sr} \). \(^9\) We assume that the matrix on the left is invertible, so that we can multiply both sides of the equation above by its inverse to obtain

\[
y_{t+1} = A y_t + B u_t + C \eta_t, \tag{6}
\]

where \( A = \tilde{G}^{-1} \tilde{A}, B = \tilde{G}^{-1} \tilde{B}, \) and \( C = \tilde{G}^{-1} \tilde{C}. \)

Following Eggertsson and Woodford (2003) and Jung, Teranishi, and Watanabe (2005), we define the one-period loss function of the central bank

\[
L(y_t, u_t) = (y_t' R y_t + u_t' Q u_t), \tag{7}
\]

where \( R \) is a \((5 \times 5)\) real positive semidefinite matrix, which is described below, and \( Q \) is a \((1 \times 1)\) real symmetric positive definite matrix. Moreover, we presume that the central bank has an infinite horizon perspective.

Consequently, the intertemporal loss function of the central bank is given as

\[
E_0 \sum_{t=0}^{\infty} \delta^t L(y_t, u_t), \tag{8}
\]

where \( E_0 \) represents the conditional expectation, given the information known at period 0 and \( \delta \in (0, 1] \) is a discount factor. The central bank minimizes the intertemporal loss function subject to (6), the non-negativity constraint on the short-term nominal interest rate \( (\hat{i}_t \geq 0), \) and an initial condition for \( z_0, \) but not for \( x_0. \)

\(^9\)Details of \( \tilde{G}, \tilde{A}, \tilde{B} \) are described in the Appendix.
The central bank conducts unconventional monetary policy in $k$ periods. We assume that the ZLBs bind from time 0 to $k$ and occur only once in the economy. Using the $k$, the intertemporal loss function of the central bank can be rewritten as

$$E_0 \sum_{t=0}^{\infty} \delta^t L(y_t, u_t) = \left[ E_0 \sum_{t=0}^{k} \delta^t (y'_t R y_t + u'_t Q u_t) + E_0 \sum_{t=k+1}^{\infty} \delta^t (y'_t R y_t + u'_t Q u_t) \right].$$  \hspace{1cm} (9)

The first term of the right hand-side of Eq. (9) is a finite horizon problem and the second term of the right hand-side is an infinite horizon problem.

### 2.2 Constrained Stackelberg solution

A constrained Stackelberg solution is described completely using the following two equations (their derivation is shown in appendix A).

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \Lambda_{\varsigma} \begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix} + \begin{bmatrix} I & 0 \\ P_{\varsigma,21} & P_{\varsigma,22} \end{bmatrix} C \eta_t,$$

and

$$x_t = \begin{bmatrix} -P_{r,22}^{-1} P_{\varsigma,21} & P_{r,22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix},$$

where $I$ is the identity matrix, $P_{\varsigma,21}$ and $P_{\varsigma,22}$ are elements of $P_{\varsigma}$, $\mu_{x,t}$ is the Lagrange multiplier vector of non-predetermined variables, and initial states, $z_0$ and $\mu_{x,0}$, are given (details of these symbols are described in Appendix C).  \hspace{1cm} (10)

The symbol, $\Lambda_{\varsigma}$, is written as

$$\Lambda_{\varsigma} = \begin{bmatrix} I & 0 \\ P_{r,21} & P_{r,22} \end{bmatrix} (A - BF) \begin{bmatrix} I & 0 \\ -P_{r,22}^{-1} P_{r,21} & P_{r,22}^{-1} \end{bmatrix},$$

if there exists no zero interest rate bound problem in the economy, and

$$\Lambda_{\varsigma} = \begin{bmatrix} I & 0 \\ P_{l,21} & P_{l,22} \end{bmatrix} A \begin{bmatrix} I & 0 \\ -P_{l,22}^{-1} P_{l,21} & P_{l,22}^{-1} \end{bmatrix},$$

if the zero interest rate bound problem exists. The symbols $\Lambda_{\varsigma} = \{r,l\}$ are stable because $z_t$ and $\mu_{x,t}$ are pre-determined. Note that $u_t = \hat{\eta}_t$ in this paper.

---

\textsuperscript{10} Eq. (11) extends readily to include the non-predetermined variable $z_t$ and measurement errors as follows.

$$\begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -P_{\varsigma,22}^{-1} P_{\varsigma,21} & P_{\varsigma,22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{x,t} \end{bmatrix} + v_t,$$

where $v_t$ is a measurement-error vector. This equation is the measurement equation of a state space model. Consequently, our method is readily applicable to estimation. Yano (2009) estimates DSGE models with ZLBs with a different approach based on the particle filtering.
3 Numerical analysis

Using Eq. (10) and (11), we can conduct Monte Carlo simulation. The algorithm of our method is described below.

Algorithm 1

1. Initialize parameters, vectors, and matrices.
2. Generate $\eta_0$.
3. Calculate Eq. (6) from Eq. (5).
4. Obtain $P_r$ and $P_l$ using the discounted Riccati equation and the discounted Lyapunov equation, respectively.
5. Simulate constrained Stackelberg paths using Eq. (10) and Eq. (11).
6. Calculate expected values of simulated paths.

To verify the effectiveness of our algorithm 1 in the case with no ZLBs, we simulate a “conventional monetary policy” using it in Figure 2. In this case, our results show that the central bank, which is conducting conventional monetary policy, controls the output gap (upper panel of Figure 2) and the inflation rate (lower panel of Figure 2) because the economy does not hit the ZLB. These results demonstrate that our method is consistent of the implications of previous studies of monetary policy and theory.

To evaluate the effects of ZIRP and expectations in the future, simulating the ZIRPs with three strategies is conducted:

- Strategy 1: A zero inflation expectation (central bank committing ZIRP and a zero inflation expectation).
- Strategy 2: A positive inflation expectation (central bank committing ZIRP and a positive inflation expectation).
- Strategy 3: A positive output gap expectation (central bank committing ZIRP and a positive output gap expectation).

In the initial states of the strategies, the initial output gap is zero and the initial inflation rate is $-0.02$ (deflation) in $y_0$.  

---

11 In Figure 2, simulated lines generated by 50 Monte Carlo simulations using algorithm 1, are drawn; in the following figures the same shall apply.

12 Details of parameters are described in Appendix C.
Figure 2: Conventional Monetary Policy.
The results of strategy 1 are shown in Figure 3, which shows that ZIRP with a zero inflation expectation has little or no influence on the economy in early periods. The upper panel confirms that the policy has little or no impact to raise the output gap. The lower panel shows that the policy has very little effect in raising inflation. These results are consistent with the experience of BOJ that ZIRP and QE, which have committed themselves to very low positive inflation, have very little influence to stimulate the Japanese economy and escape from deflation. After four or five periods, the spread of simulated lines are observed stemming from small innovations $\eta_0$. This indicates that instability of the economy exists in the late periods of strategy 1.

![Output Gap](image1.png)

![Inflation](image2.png)

Figure 3: Zero interest rate policy with zero inflation expectation.

The results of strategy 2 are shown in Figure 4. The figure confirms that ZIRP with mild inflation has weak positive influences on the economy in early periods. The difference between

---

13 The unconventional monetary policy of BOJ in the late 1990s and the 2000s includes statements that the bank will keep ZIRP and QE until the inflation rate, measured by CPI, will become greater than zero. In 2006, the bank ceased QE, nevertheless the inflation rate was around zero or showing mild deflation. Consequently, generally, it is considered that the bank was committing very low or zero inflation in the late 1990s and 2000s.
strategy 1 and 2 shows that committing mild inflation exerts positive effects on the economy and that managing inflation expectations is necessary to escape from the bounds. The upper panel shows that the policy has a weak positive impact on stimulating the economy. The lower panel shows that the power to raise the inflation rate has a very weak positive effect. These results are consistent with the experiences of the Fed in the mid-2000s that its QE and QE2 did not cause high inflation. In common with strategy 1, there exists instability of the economy in the late periods. This point is discussed below.

Figure 4: Zero interest rate policy with positive inflation expectations.

Results of Strategy 3 are presented in Figure 5. The upper panel of the figure confirms that Strategy 3 has a strong impact on the rise the output gap in the early periods. The lower panel shows that the policy has a very weak influence on raising inflation in early periods. Strategy 3 has the greatest strength to stimulate the economy of the three strategies. Unfortunately, however, the central bank practically has very few policy options to raise output gap expectations. Consequently, raising the output gap expectations might be ruled out in a feasible group of policies to escape from deflation and/or stagnation when the economy hits ZLBs. In common
with strategy 1 and 2, there exists instability of the economy in the late periods. This point is also discussed below.

Figure 5: Zero interest rate policy with positive output gap expectation.

Comparison of Figure 2 with Figure 3, 4, and 5 shows that instabilities of the economy exists, which hits the ZLBs because their simulated lines spread after the fourth or fifth period. These instabilities also reveal that the economy, which hits the ZLBs, is extremely vulnerable to exogenous shocks. Consequently, they confirm (1) vulnerability of the economy with ZLBs and (2) necessity to escape from them. Managing inflation expectations, for example inflation targeting, is crucial in zero interest rate periods.

4 Conclusions

Difficulties exist in escaping from stagnation and/or deflation if the economy hits zero lower bounds on short-term nominal interest rates because the central bank cannot stimulate the economy using rate cuts. How can those difficulties be avoided? Answering the question, we
extend a closed-looped solution of a Stackelberg problem by introducing zero lower bounds. The extension revealed that the discounted Lyapunov equation is necessary to obtain the shadow price of the economy which hits the zero lower bounds. Additionally, we emphasize that our method is consistent with the rational expectations hypothesis.

As described in this paper, we apply our method to new Keynesian models with ZLBs to evaluate the quantitative effects of zero interest rate policies with committing mild or zero inflation. Our simulation shows that committing mild inflation causes positive effects on the economy and managing inflation expectations are necessary to escape from the bounds. Additionally, it is apparent that the economy, which hits ZLBs, is vulnerable to exogenous shocks in the late periods. This indicates that there exists necessity to escape from deflation and ZLBs.

In the next step, we apply our method to extended new Keynesian models to explore the effects of fiscal policy and exchange rate intervention. Furthermore, we devise a novel estimation method based on Eq. (10) and (12) using sequential Monte Carlo methods proposed by Kitagawa (1996) and Gordon, Salmond, and Smith (1993). Moreover, we are investigating a robust control version of our method based on Hansen and Sargent (2003) and Hansen and Sargent (2007).

5 Acknowledgments

The author would like to thank Yasushi Okada, Takeshi Kudo, participants of 2013 Japanese Economic Association Spring Meeting, and the members of ESRI DSGE team for their helpful comments. This work was supported by Special Research Program of Komazawa University (2011) and JSPS KAKENHI Grant Number 50584497.

A Constrained Stackelberg problem

A.1 Linear-quadratic regulator with zero lower bounds

First, we set the linear-quadratic regulator with zero lower bounds and derive the optimal decision rule of it. 14

We define the one-period social loss function as

\[ L(y_t, u_t) = (y_t'Ry_t + u_t'Q u_t), \]  

where \( y_t \) is an \((n \times 1)\) vector of state variables, \( u_t \) is an \((m \times 1)\) vector of control variables, \( R \) is an \((n \times n)\) real positive semidefinite matrix, \( Q \) is an \((m \times m)\) real symmetric positive definite matrix, Moreover, we assume that the central bank has an infinite horizon perspective.

14See Hanson (1964).
Therefore, the intertemporal social loss function is given as
\[
\min_{\{y_{t+1}, u_t\}} E_0 \left[ \sum_{t=0}^{\infty} \delta^t L(y_t, u_t) \right].
\] (16)

The linear-quadratic optimal regulator problem is to minimize Eq. (16) subject to a fixed initial condition for \( y_0 \) and the law of motion \( y_{t+1} = Ay_t + Bu_t + C\eta_t \), where \( A \) is an \((n \times n)\) matrix, \( B \) is an \((n \times m)\) matrix, \( C \) is an \((n \times k)\) matrix, \( \eta_t \) is a \((k \times 1)\) vector of exogenous shocks, and \( \delta \in (0, 1] \) is a discount factor.

We presume that there exists a Zero Lower Bound (ZLB) on the vector of control variables.

Assumption A.1.
\[ u_t \geq 0. \] (17)

As we shall discuss below, the ZLB makes it a practical issue to control the state vector because we cannot determine the optimal policy rule when the control vector hits the ZLB. Moreover, we assume that the ZLBs bind from time 0 to \( k \) and that they occur only once in the economy.

Considering the ZLBs, we construct the Bellman equation of our problem:
\[
y_t' P_\zeta y_t = \min_{\{y_{t+1}, u_t\}} E_t \left[ y_t' P_\zeta y_t + u_t'Qu_t + \delta y_{t+1}' P_\zeta y_{t+1} + \lambda_t u_t \right].
\] (18)
subject to the law of motion \( y_{t+1} = Ay_t + Bu_t + C\eta_t \) and \( u_t \geq 0 \), where \( \lambda_t \) is a multiplier vector. The optimal value function of the linear regulator has the form \( v(y) = -y'P\zeta y \), where \( P_\zeta \) is defined as
\[
P_\zeta = \begin{cases} 
P_l & \text{when the economy hits the ZLBs} \\
P_r & \text{otherwise} 
\end{cases}
\] (19)
The determination of \( P_\zeta \) is discussed below.

Next, using the law of motion, the Bellman equation can be rewritten as
\[
y_t' P_\zeta y_t = \min_{u_t} \left[ y_t' Ry_t + u_t'Qu_t + \delta (Ay_t + Bu_t + C\eta_t)' P_\zeta (Ay_t + Bu_t + C\eta_t) + \lambda_t u_t \right].
\] (20)
The first-order condition with respect to \( u_t \) is
\[
(Q + \delta B'P_\zeta B)u_t + \delta B'P_\zeta Ay_t + \lambda_t = 0
\] (21)
By the Karush–Kuhn–Tucker theorem, \(^{15}\) the following equation holds:
\[
\lambda_t u_t = 0, \quad \text{(complementary slackness).}
\] (22)

\(^{15}\)See Carter (2001).
Because Eq. (22) has complementary slackness, it is necessary to consider case 1 ($\lambda_t = 0$ and $u_t > 0$) and case 2 ($\lambda_t > 0$ and $u_t = 0$) to derive the optimal decision rule of the linear-quadratic regulator.

In case 1 ($\lambda_t = 0$ and $u_t > 0$), Eq. (21) is rewritten as

\[(Q + \delta B' P r B)u_t + \delta B' P r Ay_t = 0.\]

(23)

Immediately, it leads us to the standard optimal decision rule

\[u_t = -F y_t,\]

(24)

where $F = \delta (Q + \delta B' P B)^{-1} B' P A$.

In case 2 ($\lambda_t > 0$ and $u_t = 0$), there does not exist the optimal decision rule because $u_t = 0$.

As alternated, there is a rule to determine the multiplier $\lambda_t$ as shown below.

\[\lambda_t = -\delta B' P_A y_t\]

(25)

### A.2 Shadow prices of the problem

Second, we derive the shadow price of the problem.

The Lagrangian of the problem is

\[L = E_0 \sum_{t=0}^{\infty} \delta^t \left\{ L(y_t, u_t) + 2\delta \mu_t(Ay_t + Bu_t + C\eta_t - y_{t+1}) + 2\lambda_t u_t \right\},\]

(26)

where $\mu_t$ is a Lagrange multiplier vector.\(^\text{16}\) The first-order conditions with respect to $u_t$ and $y_t$ are respectively shown below.

\[u_t : Qu_t + \delta B' \mu_{t+1} + \lambda_t = 0\]

(27)

\[y_t : \mu_t = R y_t + \delta A' \mu_{t+1}\]

(28)

By the Karush–Kuhn–Tucker theorem, the following equation holds:

\[\lambda_t u_t = 0, \quad \text{(complementary slackness).}\]

(29)

In case 1 ($\lambda_t = 0$ and $u_t > 0$), standard textbooks on control theory (e.g., Ljungqvist and Sargent (2004)) teach us that a stabilizing solution satisfies the equation shown below.

\[\mu_t = P_r y_t\]

(30)

In case 2 ($\lambda_t > 0$ and $u_t = 0$), the following equation holds.

\[\lambda_t = -\delta B' \mu_{t+1}\]

(31)

\(^{16}\)For simplicity, we drop conditional expectations $E_t$ when this is not a source of confusion below.
Combining Eq. (25) and (31), we obtain

$$\mu_{t+1} = P_tAy_t$$

(32)

Therefore, the following equation holds

$$\mu_t = P_ty_t$$

(33)

because $y_{t+1} = Ay_t$.

Combining Eq. (30) and (33), finally, we obtain the shadow price of our problem

$$\mu_t = P_ty_t$$

(34)

A.3 Stabilizing solution

In case 1 ($\lambda_t = 0$ and $u_t > 0$), standard textbooks on control theory (e.g., Ljungqvist and Sargent (2004)) teach us that the matrix $P_r$, which is a realization of $P_s$, satisfies the following discounted Riccati equation.

$$P_r = R + \delta A'P_rA - \delta^2 A'P_rB(Q + \delta B'P_rB)^{-1}B'P_rA$$

(35)

In case 2 ($\lambda_t > 0$ and $u_t = 0$), we must derive the new equation by which the matrix $P_l$, which is a realization of $P_s$, must be satisfied because there does not exist an optimal decision rule. The law of motion can be rewritten as

$$y_{t+1} = Ay_t$$

(36)

because $u_t = 0$. Premultiplying Eq. (36) by $\delta A'P_l$ gives

$$\delta A'P_l y_{t+1} = \delta A'P_lAy_t$$

(37)

Combining Eq. (28) and Eq. (33),

$$\delta A'P_l y_{t+1} = -Ry_t + P_ly_t$$

(38)

The right side of Eq. (37) agrees with the right side of Eq. (38) for initial value $y_0$. Therefore, in case 2, the following discounted Lyapunov equation must hold:

$$P_l = R + \delta A'P_lA$$

(39)

Eq. (39) is the major contribution of this paper because it defines the shadow price of the linear-quadratic optimal regulator when ZLBs holds. The equation, which we derive here, is consistent with the recent development of constrained model predictive control, for example,
Rawlings and Muske (1993). We emphasize that adopting Eq. (39) is necessary to simulate ZIRP and that using Eq. (35) causes the misspecification of the Stackelberg problem when the economy hits ZLBs.

A.4 Constrained Stackelberg solution

In this subsection, the closed-loop solution of a constrained Stackelberg problem is described. Since the seminal works of Kydland and Prescott (1977), Calvo (1978), Barro and Gordon (1983a), and Barro and Gordon (1983b), it has become widely known that the open-loop solutions of the Stackelberg problem are time-inconsistent. Ljungqvist and Sargent (2004), however, derive the closed-loop solution of the problem using the linear-quadratic optimal regulator.

First, we designate 
\[ z_t \] is an \((n_z \times 1)\) vector of predetermined variables, \[ x_t \] is an \((n_x \times 1)\) vector of non-predetermined variables, \[ u_t \] is an \((m \times 1)\) vector of government instruments, \[ y_t = [z_t, x_t]^T, \] and \( n = n_z + n_x \). Second, a structural linear rational expectations model of the economy is given as

\[ \tilde{G} y_{t+1} = \tilde{A} y_t + \tilde{B} u_t + \tilde{C} \tilde{\eta}_t, \]  
(40)

where \( y_t \) is an endogenous-variable vector, \( \tilde{G}, \tilde{A}, \tilde{B}, \tilde{C} \), are matrices, and \( \tilde{\eta}_t \sim N(0, \Sigma_{\eta_t}) \) with the \((k \times k)\) identity matrix \( \Sigma_{\eta_t} \). We assume that the matrix on the left is invertible, so that we can multiply both sides of the above equation by its inverse to obtain

\[ y_{t+1} = Ay_t + Bu_t + C \eta_t, \]  
(41)

where \( A = \tilde{G}^{-1} \tilde{A}, B = \tilde{G}^{-1} \tilde{B}, C = \tilde{G}^{-1} \tilde{C} \), and \( \eta_t = \tilde{G}^{-1} \tilde{\eta}_t \). We define the one-period loss function of the central bank as

\[ L(y_t, u_t) = (y_t' R y_t + u_t' Q u_t), \]  
(42)

where \( R \) is a \((n \times n)\) real positive semidefinite matrix and \( Q \) is a \((m \times m)\) real symmetric positive definite matrix. Consequently, the intertemporal loss function of the central bank is given as

\[ E_0 \sum_{t=0}^{\infty} \delta^t L(y_t, u_t) \]  
(43)

where \( E_0 \) is the conditional expectation, given the information known at period 0 and \( \delta \in (0, 1] \) is a discount factor. The central bank minimizes the intertemporal loss function subject to (41) and an initial condition for \( z_0 \), but not for \( x_0 \).

---

17 Constrained model predictive control is a rapidly developing area in process control and related studies of them are surveyed in Mayne, Rawlings, Rao, and Scokaert (2000), Maciejowski (2002), and Rawlings and Mayne (2009).

18 The description of this appendix is based on chapter 18, Ljungqvist and Sargent (2004).
We partition the Lagrange multiplier vector of the Lagrangian (26),

\[
\mu_t = \begin{bmatrix}
\mu_{z,t} \\
\mu_{x,t}
\end{bmatrix}
\]

where \( \mu_{z,t} \) is the Lagrange multiplier vector of predetermined variables and \( \mu_{x,t} \) is the Lagrange multiplier vector of non-predetermined variables. For the constrained Stackelberg problem, the first \( n_z \) elements of \( y_t \) are predetermined but the remaining components are non-predetermined, whereas the first \( n_z \) elements of \( \mu_t \) are non-predetermined, but the remaining components are predetermined. Following Ljungqvist and Sargent (2004), we can rewrite Eq. (34) as follows.

\[
\begin{bmatrix}
\mu_{z,t} \\
\mu_{x,t}
\end{bmatrix} = \begin{bmatrix}
P_{\varsigma,11} & P_{\varsigma,12} \\
P_{\varsigma,21} & P_{\varsigma,22}
\end{bmatrix} \begin{bmatrix}
z_t \\
x_t
\end{bmatrix},
\]

Therein, \( P_{\varsigma,11}, P_{\varsigma,12}, P_{\varsigma,21}, \) and \( P_{\varsigma,22} \) are the elements of \( P_{\varsigma} \). Therefore, we solve Eq. (45) for \( x_t \) in terms of \( z_t \) and \( \mu_{x,t} \):

\[
x_t = -P_{\varsigma,22}^{-1}P_{\varsigma,21}z_t + P_{\varsigma,22}^{-1}\mu_{x,t}.
\]

Therefore, we can write the following

\[
y_t = \begin{bmatrix}
z_t \\
x_t
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
-P_{\varsigma,22}^{-1}P_{\varsigma,21} & P_{\varsigma,22}^{-1}
\end{bmatrix} \begin{bmatrix}
z_t \\
\mu_{x,t}
\end{bmatrix}.
\]

and

\[
\mu_{x,t} = \begin{bmatrix} P_{\varsigma,21} & P_{\varsigma,22} \end{bmatrix} y_t,
\]

where \( I \) is the identity matrix.

In case 1, using Eq. (47), the optimal policy function, Eq. (24), is rewritten as

\[
u_t = -F \begin{bmatrix}
I & 0 \\
-P_{\varsigma,22}^{-1}P_{\varsigma,21} & P_{\varsigma,22}^{-1}
\end{bmatrix} \begin{bmatrix}
z_t \\
\mu_{x,t}
\end{bmatrix}.
\]

Using Eq. (41), (47), and (49), the following equation is obtained.

\[
\begin{bmatrix}
z_{t+1} \\
\mu_{x,t+1}
\end{bmatrix} = \begin{bmatrix}
P_{r,21} & P_{r,22}
\end{bmatrix} (A - BF) \begin{bmatrix}
z_t \\
\mu_{x,t}
\end{bmatrix} + \begin{bmatrix}
P_{r,21} & P_{r,22}
\end{bmatrix} C y_{t+1} C' y_{t+1}.
\]

In case 2, the vector of control variables hits a zero lower bound:

\[
u_t = 0
\]

\[
\begin{bmatrix}
I & 0 \\
P_{\varsigma,21} & P_{\varsigma,22}
\end{bmatrix} \begin{bmatrix}
I & 0 \\
-P_{\varsigma,22}^{-1}P_{\varsigma,21} & P_{\varsigma,22}^{-1}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]
Using Eq. (41), (47), and (51), the following equation is obtained.

\[
\begin{bmatrix}
z_{t+1} \\
\mu_{x,t+1}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
P_{t,21} & P_{t,22}
\end{bmatrix} A \begin{bmatrix}
I & 0 \\
-P_{t,22}^{-1}P_{t,21} & P_{t,22}^{-1}
\end{bmatrix} \begin{bmatrix}
z_t \\
\mu_{x,t}
\end{bmatrix} + \begin{bmatrix}
I & 0 \\
P_{t,21} & P_{t,22}
\end{bmatrix} C \eta_t.
\]

Combining Eq. (46), (50) and (52), Finally, we obtain Eq. (10) and (11), which are the closed-loop solutions of a constrained Stackelberg problem for the central bank.

\section{Zero lower bounds and Lyapunov stability}

In this section, we prove the Lyapunov stability of Eq. (10) when the economy hits the ZLBs. In our specification (appendix D), the matrix \( R \) is defined as follows.

\[
R = \begin{bmatrix}
R_{11} & 0 \\
0 & 0
\end{bmatrix}
\]

Moreover, we denote

\[
y_{\mu,t} = \begin{bmatrix}
z_t \\
\mu_{x,t}
\end{bmatrix}.
\]

Using Eq. (46),

\[
y_{\mu,t}'Ry_t = \begin{bmatrix}
z_t' \\
x_t'
\end{bmatrix} \begin{bmatrix}
R_{11} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
z_t \\
x_t
\end{bmatrix} = y_{\mu,t}'Ry_{\mu,t}
\]

The value function of the economy which hits the ZLBs is defined as follows.

\[
V_\mu(k) = \sum_{t=1}^{\infty} \delta^t L_\mu(y_{\mu,k+t}) , \quad L_\mu(y_{\mu,t}) = y_{\mu,t}'Ry_{\mu,t}
\]

subject to Eq. (10) and \( \Lambda_t \). Furthermore, we suppose that \( V_\mu^*(k) \) is optimal value function of this problem. The vectors \( z_t \) and \( x_t \) converge to steady states \( \bar{z} \) and \( \bar{x} \), respectively, because \( \Lambda_t \) is stable. Without loss of generality, we can assume the steady states are equivalent to zero.

From the Lapunov theorem, the following equation holds. \(^{21}\)

\[
V_\mu^*(k) \geq V_\mu^*(k + 1).
\]

Therefore, our problem is Lapunov stable.

\(^{20}\)In Hirose (2007), he finds that a sunspot equilibrium has a substantial effects on both output and inflation in the economy which hits ZLBs.

\(^{21}\)See Rawlings and Mayne (2009).
C Nonlinear Taylor rule

Using our method, we derive the nonlinear Taylor rule (the Taylor rule with a zero lower bound) in this section. The concept of our derivation is similar to that of Giannoni and Woodford (2003), building on the earlier work by Svensson and Woodford (2004). They derive an optimal monetary policy rule in the case of a dynamic linear rational-expectations model and a linear-quadratic objective function for policy. However, our main difference from them is that the rule has a ZLB.

In case 1, Eq. (10) and (24) can be represented compactly:

\[
\begin{bmatrix}
    z_{t+1} \\
    \mu_{x,t+1}
\end{bmatrix} =
\begin{bmatrix}
    A_{11,r} & A_{12,r} \\
    A_{21,r} & A_{22,r}
\end{bmatrix}
\begin{bmatrix}
    z_t \\
    \mu_{x,t}
\end{bmatrix} +
\begin{bmatrix}
    C'_{11} & C'_{12} \\
    C'_{21} & C'_{22}
\end{bmatrix}
\begin{bmatrix}
    \eta_{x,t} \\
    \eta_{z,t}
\end{bmatrix},
\]

and

\[
u_t = f_{11} z_t + f_{12} \mu_{x,t},
\]

Eq. (60) can be rewritten as

\[
\mu_{x,t} = f_{11}^{-1} (u_t - f_{11} z_t),
\]

where \( f_{11}^{-1} \) is the generalized inverse of \( f_{11} \). Substituting Eq. (61) into the lower side of Eq. (59), we can obtain the Taylor rule as shown below.

\[
u_t = f_{12} A_{22,r} f_{12}^{-1} u_{t-1} + f_{11} z_t
\]

\[
+ f_{12} (A_{21,r} z_t - A_{21,r} f_{12}^{-1} f_{12}^1 z_{t-1} + C'_{21} \eta_{z,t-1} + C'_{22} \eta_{x,t-1})
\]

Using Eq. (51) and (62), and the elements of \( z_t \) and \( u_t \). Finally, we obtain the nonlinear Taylor rule as follows.

\[
\hat{i}_t = \max \left[ 0, \rho_{u11} \hat{i}_{t-1} + \rho_{u2,y} \hat{y}_t + \rho_{u2,\pi} \hat{\pi}_t + \rho_{u2,\pi' \pi} \hat{\pi}_t + \rho_{u13,\pi} \hat{\pi}_t -1 \rho_{u3,y} \hat{y}_{t-1} + \rho_{u3,\pi} \hat{\pi}_{t-1} + \rho_{u3,\pi' \pi} \hat{\pi}_{t-1} + C'_{21} \eta_{z,t-1} + C'_{22} \eta_{x,t-1} \right],
\]

Therein, \( \rho_{u1} \) is \( f_{12} A_{22,r} f_{12}^{-1} \), \( \rho_{u2,y} \), \( \rho_{u2,\pi} \), and \( \rho_{u2,\pi' \pi} \) respectively denote the first, second, and third elements of \( f_{11} \), and \( \rho_{u3,y} \), \( \rho_{u3,\pi} \), and \( \rho_{u3,\pi' \pi} \) respectively represent the first, second, and third elements of \( f_{12} (A_{21,r} z_t - A_{21,r} f_{12}^{-1} f_{12}^1) \).

D Matrices and calibration

For convenience, we designate \( z_t = [\hat{Y}_t, \hat{\pi}_t, \hat{\pi}_t^n] \) as the vector of predetermined variables, \( x_t = [E_t \hat{Y}_{t+1}, E_t \hat{\pi}_{t+1}] \) as the vector of non-predetermined variables, \( y_t = [z_t, x_t]^T \), and \( u_t = [\hat{i}_t] \geq 0 \).
The matrices of Equation (5) are

\[ \tilde{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -\frac{1}{1+h} & -\frac{1}{1+h} & -\frac{1}{1+h} \\ -\kappa & 1 & 0 & 0 & -\frac{\delta}{1+h} \end{bmatrix}, \]

\[ \tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \rho_n & 0 & 0 \\ \frac{h}{1+h} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1+h} & 0 & 0 & 0 \end{bmatrix}, \]

\[ \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{1+h} \\ 0 \end{bmatrix}, \]

\[ \eta = \begin{bmatrix} 0 \\ 0 \\ \eta_t^\gamma \\ \eta_t^\alpha \end{bmatrix}, \]

and

\[ \tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\( \Sigma_{sr}^2 \) is the \((n_a \times n_a)\) identity matrix.
The matrices of the loss function (42) are

\[
R = \begin{bmatrix}
R_1 & 0 & 0 & 0 & 0 \\
0 & R_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

(65)

\[
Q = [Q_1],
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta)</td>
<td>0.3</td>
</tr>
<tr>
<td>(\beta)</td>
<td>0.00</td>
</tr>
<tr>
<td>(h)</td>
<td>0.7</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>0.024</td>
</tr>
<tr>
<td>(\rho_n)</td>
<td>0.011</td>
</tr>
<tr>
<td>(R_1 = R_2)</td>
<td>1</td>
</tr>
<tr>
<td>(Q_1)</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Calibrated Parameters.

In simulations of strategies 1, 2, and 3, we set the respective initial state variables \(y_0 = [0.0, -0.02, 0, 0, 0]\), \(y_0 = [0.0, -0.02, 0, 0, 0, 0.02]\), and \(y_0 = [0.0, -0.02, 0, 0.02, 0.02, 0.0]\). Other calibrated parameters are presented in Table 1.

References


